Essentials of *k*-essence

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We recently introduced the concept of ''*k*-essence'' as a dynamical solution for explaining naturally why the universe has entered an epoch of accelerated expansion at a late stage of its evolution. The solution avoids fine-tuning of parameters and anthropic arguments. Instead, *k*-essence is based on the idea of a dynamical attractor solution which causes it to act as a cosmological constant only at the onset of matter domination. Consequently, *k*-essence overtakes the matter density and induces cosmic acceleration at about the present epoch. In this paper, we present the basic theory of *k*-essence and dynamical attractors based on evolving scalar fields with nonlinear kinetic energy terms in the action. We present guidelines for constructing concrete examples and show that there are two classes of solutions, one in which cosmic acceleration continues forever and one in which the acceleration has finite duration.

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I. INTRODUCTION

A concordance of cosmological observation [1] of large-scale structure, the cosmic microwave background anisotropy and type IA supernovas at deep redshift suggests that the matter density of the universe comprises about one-third of the critical value expected for a flat universe. The missing two-thirds is due to an exotic dark energy component with negative pressure that causes the Hubble expansion to accelerate today. One candidate for such a component is a cosmological constant (Λ) or vacuum density. Another possibility is a dynamical component whose energy density and spatial distribution evolve with time, as is the case for quintessence [2] or, as explored herein, for *k*-essence [3].

A key challenge for theoretical physics is to address the cosmic coincidence problem: why does the dark energy component have a tiny energy density $[\mathcal{O}(\mathrm{meV}^4)]$ compared to the naive expectation based on quantum field theory and why does cosmic acceleration begin at such a late stage in the evolution of the universe. Most dark energy candidates (such as the cosmological constant) require extraordinary fine-tuning of the initial energy density to a value 100 orders of magnitude or more smaller than the initial matter-energy density. Proponents of anthropic models [7] often pose the problem as the following: why should the acceleration begin shortly after structure forms in the universe and sentient beings evolve? If the dark energy component consists of vacuum density (Λ) or quintessence [2] in the forms that have been discussed in the literature to date, the answer is either pure coincidence or the anthropic principle.

The purpose of introducing *k*-essence is to provide a dynamical explanation which does not require the fine-tuning of initial conditions or mass parameters and which is decidedly non-anthropic. In this scenario, cosmic acceleration and human evolution are related because both phenomena are linked to the onset of matter domination. The *k*-essence component has the property that it only behaves as a negative pressure component after matter-radiation equality, so that it can only overtake the matter density and induce cosmic acceleration after the matter has dominated the universe for some period, at about the present epoch. And, of course, human evolution is linked to matter domination because the formation of planets, stars, galaxies and large-scale structure only occurs during this period. A further property of *k*-essence is that, because of the dynamical attractor behavior, cosmic evolution is insensitive to initial conditions.

The existence of attractor solutions is reminiscent of quintessence models based on evolving scalar fields with exponential [4] ''tracker'' [5,6] potentials. In these models, an attractor solution causes the energy density in the scalar field to track the equation of state of the dominant energy component, be it radiation or matter. An advantage is that the cosmic evolution is insensitive to the initial energy density of the quintessence field, and, for many models, the scenario can begin with the most natural possibility, equipartition initial conditions. (For the case of vacuum energy or cosmological constant, the vacuum energy must be set 120 orders of magnitude less than the initial matter-radiation density.) However, as long as the field tracks any equation of state, it cannot overtake the matter density and induce cosmic acceleration. Indeed, for a purely exponential potential, the field never overtakes the matter density and dominates the universe. Hence, this is an unacceptable candidate for the dark energy component. In tracker models, the problem is addressed because the curvature of the potential ultimately dips to a critically small value once the field passes a particular value $\bar{Q}$ such that the field Q becomes frozen and begins to act like a cosmological constant. The value of the potential energy density at $Q=\bar{Q}$ determines when quintessence overtakes the matter density and cosmic acceleration begins. The overall scale of the potential must be finely adjusted in order for the component to overtake the matter density at the present epoch. So while tracker models allow equipartition initial conditions, they require the same fine-tuning as models with cosmological constant.

The distinctive feature of the *k*-essence models we consider is that *k*-essence only tracks the equation of state of the background during the radiation-dominated epoch. A tracking solution during the matter-dominated epoch is physically forbidden. Instead, at the onset of matter domination, the *k*-essence field energy density ε drops several orders of magnitude as the field approaches a new attractor solution in which it acts as a cosmological constant with pressure p approximately equal to $-\varepsilon$. That is, the equation of state, $w \equiv p/\varepsilon$, is nearly -1. The *k*-essence energy density catches up and overtakes the matter density, typically several 10^9's of years after matter domination, driving the universe into a period of cosmic acceleration. As it overtakes the energy density of the universe, it begins to approach yet another attractor solution which, depending on the details, may correspond to an accelerating universe with $w<-1/3$ or a decelerating or even dust-like solution with $-1/3<w\leqslant 0$. In this scenario, we observe cosmic acceleration today because the time for human evolution and the time for *k*-essence to overtake the matter density are both several 10^9's of years due to independent but predictive dynamical reasons.

The *k*-essence models which we have found rely on dynamical attractor properties of scalar fields with non-linear kinetic energy terms in the action, models which are unfamiliar to most particle physicists and cosmologists. Some of the concepts were first introduced to develop an alternative inflationary model known as *k*-inflation [8]. Chiba *et al.* [9] have discussed kinetic-energy driven quintessence, but in a different context which does entail dynamical attractors and the resolution of the cosmic coincidence problem. In this paper, we present a thorough, pedagogical study of dynamical attractor behavior and the application to present-day cosmic acceleration. The paper is organized as follows: In Sec. II we derive the basic equations describing the dynamics of a universe filled by matter, radiation and *k*-essence. In Sec. III, we classify the possible attractor solutions for *k*-essence. In some cases, the attractor solution causes *k*-essence to mimic the equation of state of the background energy density; we refer to this as a *tracker* solution. In other cases, *k*-essence mimics a cosmological constant, quintessence or dust, without depending on the presence of any additional cosmic energy density. In Sec. IV, we show how these principles can be used to control how *k*-essence travels through a series of attractor solutions as the universe evolves beginning from general initial conditions. In particular, we show how *k*-essence can transform automatically into an effective cosmological constant at the onset of matter domination, as is desired to explain naturally the present-day cosmic acceleration. In Sec. V, we show how to utilize these concepts to design model Lagrangians. We explore two illustrative examples. In one case, the future evolution of *k*-essence causes the universe to accelerate forever. In the other case, *k*-essence ultimately approaches an equation of state corresponding to pressureless dust, and the universe returns to a decelerating phase.

II. BASICS OF *k*-ESSENCE

The attractor behavior required for avoiding the cosmic coincidence problem can be obtained in models with non-standard (non-linear) kinetic energy terms. In string and supergravity theories, non-standard kinetic terms appear generically in the effective action describing the massless scalar degrees of freedom. Normally, the non-linear terms are ignored because they are presumed to be small and irrelevant. This is a reasonable expectation since the Hubble expansion damps the kinetic energy density over time. However, one case in which the non-linear terms cannot be ignored is if there is an attractor solution which forces the non-linear terms to remain non-negligible. This is precisely what is being considered here. Hence, we wish to emphasize that *k*-essence models are constructed from building blocks that are common to most quantum field theories and, then, utilize dynamical attractor behavior (which often arises in models with non-linear kinetic energy) to produce novel cosmological models.

Restricting our attention to a single field, the action generically may be expressed (perhaps after conformal transformation and field redefinition) as

$$S_\varphi=\int d^4x\sqrt{-g}\left[-\frac{R}{6}+p(\varphi,X)\right], \tag{1}$$

where we use units such that $8\pi G/3=1$ and

$$X=\frac{1}{2}(\nabla\varphi)^2. \tag{2}$$

The Lagrangian p depends on the specific particle theory model. In this paper, we consider only factorizable Lagrangians of the form

$$p=K(\varphi)\tilde{p}(X), \tag{3}$$

where we assume that $K(\varphi)>0$.

Lagrangians of this type are general enough to accommodate slow-roll, power-law and pole-like inflation, and they also appear rather naturally in the effective action of string theory. For small X, one can have $\tilde{p}(X)=\text{const}+X+\mathcal{O}(X^2)$. Ignoring quadratic and higher order terms, the theory corresponds (after field redefinition) to an ordinary scalar field with some potential. Normally, higher order kinetic energy terms are ignored under the assumption that they are small, but the attractor solutions considered here ensure that the non-linear terms remain non-negligible throughout cosmic history. The scalar field for which these higher order kinetic terms play an essential role we call, for brevity, *k*-essence.

To describe the behavior of the scalar field it is convenient to use a perfect fluid analogy. The role of the pressure is played by the Lagrangian p itself, while the energy density is given by [8]

$$\varepsilon=K(\varphi)[2X\tilde{p}_{,X}(X)-\tilde{p}(X)] \tag{4}$$

$$\equiv K(\varphi)\tilde{\epsilon}(X), \tag{5}$$

where $\ldots_{,X}$ denotes a partial derivative with respect to X. The ratio of pressure to energy density, which we call, for brevity, the *k*-essence equation of state,

$$w_k \equiv \frac{p}{\varepsilon} = \frac{\tilde{p}}{\tilde{\epsilon}} = \frac{\tilde{p}}{2X\tilde{p}_{,X} - \tilde{p}}, \tag{6}$$

does not depend on the function $K(\varphi)$. For a ''standard'' kinetic term, $p = X$, in the case when there is no potential, the equation of state is $w_k = 1$. However, for a general choice of p it is easy to get any value of w_k. Notice that $w_k < -1$ does not imply necessarily the instability of the fluid with respect to small wavelength perturbations. The *effective* ''speed of sound,'' c_S, which determines the propagation of perturbations in the *k*-essence component is [10]

$$c_S^2 = \frac{p_{,X}}{\varepsilon_{,X}} = \frac{\tilde{p}_{,X}}{\tilde{\epsilon}_{,X}}, \tag{7}$$

and it can be positive for any w_k. For instance, the effective speed of sound, defined to be the coefficient of the momentum-squared term in the perturbation equation for the scalar field, is always equal to 1 for quintessence models with canonical kinetic energy, while the equation of state w can be rather arbitrary here.

We want to study the evolution of a universe filled by *k*-essence (labeled in the equations below by k) and matter radiation (labeled by m in cases where we refer generically to the dominant matter-radiation component, be it dust-like or radiation, or by d or r if we refer specifically to the dust-like or radiation component, respectively). There is increasing evidence that the total energy density of the universe is equal to the critical value, [1,11] and, hence, we will consider a flat universe only. In that case the equation for the scale factor a takes the form

$$H^2 \equiv \dot{N}^2 = \varepsilon_m + \varepsilon_k, \tag{8}$$

where an overdot denotes a derivative with respect to physical time t and we introduced the number of e-foldings, $N = \log a$. This equation has to be supplemented with equations for ε_m and ε_k. These are the energy conservation equations for each component j:

$$\frac{d\varepsilon_j}{dN} = -3\varepsilon_j(1 + w_j), \tag{9}$$

where w_j is the equation of state for the appropriate matter-radiation or *k*-essence component. Considering a homogeneous field φ and substituting the expression for the energy density (4) into the appropriate equation (9) for *k*-essence, one gets

$$\frac{dX}{dN} = -\frac{\tilde{\epsilon}}{\tilde{\epsilon}_{,X}}\left[3(1 + w_k) + \sigma\frac{K_{,\varphi}}{K}\frac{\sqrt{2X}}{H}\right], \tag{10}$$

where w_k is given by Eq. (6), $\sigma \equiv \mathrm{sgn}(d\varphi/dN)$ and the Hubble constant is given by Eq. (8).

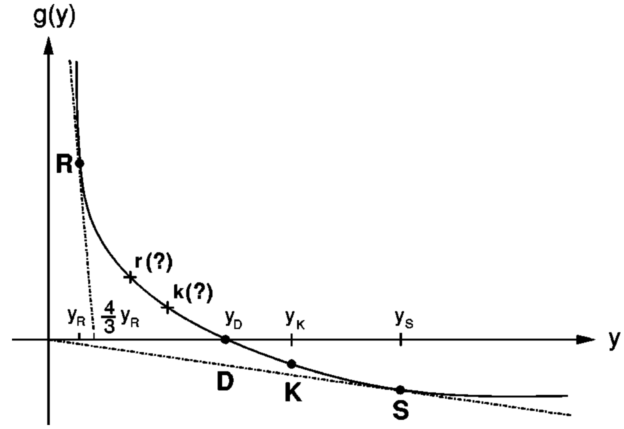

FIG. 1. A sample function $g(y)$. Boldface letters denote the corresponding attractors; their positions are given on the y axis. The tangent to the curve at a radiation tracker, such as **R**, goes through $4y_R/3$, whereas the tangent to the curve at the de Sitter point **S** goes through the origin.

We will consider functions $\tilde{p}(X)$ that increase monotonically with X. They should satisfy further restrictions, which follow from the requirements of positivity of the energy density,

$$\tilde{\epsilon} = 2X\tilde{p}_{,X} - \tilde{p} > 0, \tag{11}$$

and stability of the *k*-essence background, $c_S^2 > 0$, implying

$$\tilde{\epsilon}_{,X} = 2X\tilde{p}_{,XX} + \tilde{p}_{,X} > 0. \tag{12}$$

For designing models and visualizing constraints, it is helpful to re-express $\tilde{p}$ as $\tilde{p} = g(y)/y$ and consider it as a function of the new variable $y = X^{-1/2}$. The pressure of the *k*-essence component is, therefore,

$$p = K(\varphi)g(y)/y; \tag{13}$$

the equation of state and the effective sound speed are, correspondingly,

$$w_k = \frac{p}{\varepsilon} = -\frac{g}{yg'}, \quad c_S^2 = \frac{(g - g'y)}{g''y^2} \tag{14}$$

and the restrictions, Eqs. (11) and (12), take the very simple form

$$\tilde{\epsilon} = -g' > 0, \quad \tilde{\epsilon}_{,X} = \frac{1}{2}y^3 g'' > 0, \tag{15}$$

where the primes denote a derivative with respect to y. These conditions just mean that g should be a decreasing convex function of $y = X^{-1/2}$. A generic function which satisfies these restriction is shown in Fig. 1. Taking into account that $H = \sqrt{\varepsilon_{tot}} = \sqrt{\varepsilon_m + \varepsilon_k}$ and $\varepsilon_k = K(\varphi)\tilde{\epsilon}(y) = -Kg'(y)$ one can rewrite Eq. (10) in terms of the new variables as

$$\frac{dy}{dN}=\frac{3}{2}\frac{[w_k(y)-1]}{r'(y)}\left[r(y)+\sigma\frac{K_{,\varphi}}{2K^{3/2}}\sqrt{\frac{\varepsilon_k}{\varepsilon_{tot}}}\right], \quad (16)$$

where

$$r(y)\equiv\left(-\frac{9}{8}g'\right)^{1/2}y(1+w_k)=\frac{3}{2\sqrt{2}}\frac{(g-g'y)}{\sqrt{-g'}} \quad (17)$$

is a function which, as we will see later, is critical for the attractor properties of *k*-essence.

III. CLASSIFICATION OF TRACKER AND ATTRACTOR SOLUTIONS

The attractor solutions for *k*-essence can be divided into two classes. In one class, *k*-essence mimics the equation of state of the matter-radiation component in the universe. We refer to these as *trackers* because the cosmic evolution of *k*-essence follows the track of another energy component. The second class of attractors consists of cases where *k*-essence is drawn towards an equation of state which is different from matter or radiation. These attractors are important in the limits where *k*-essence is either a negligibly small or an overwhelming large fraction of the total energy density. The types of attractors available at any given moment in cosmic history depend on whether the universe is radiation or matter dominated. For all types of attractors, there is an associated basin of attraction, a set of initial conditions which evolve towards the attractor.

In the presence of a matter background (dust or radiation) component with constant equation of state w_m, Eq. (16) can have tracking solutions for which the *k*-essence equation of state equals w_m. To reveal when it can happen and to find these solutions explicitly we just need to note that if such solutions exist, they have to be generically of the form $y(N)=y_m=$const, where y_m satisfies the equation

$$w_k(y_m)\equiv-\left.\frac{g}{yg'}\right|_{y=y_m}=w_m. \quad (18)$$

Substituting this ansatz into Eq. (16) and noting that the ratio $\varepsilon_k/\varepsilon_{tot}$ should stay constant during the tracking stage, we see that $y(N)=y_m$ can be a solution of Eq. (16), only if $K(\varphi)=\text{const}/\varphi^2$ and, therefore, for simplicity, we consider from now on only scalar fields with Lagrangian

$$L=\frac{g(y)}{\varphi^2 y}. \quad (19)$$

It is worth noting that this kind of dependence on a scalar field occurs in the string tree-level effective action when expressed in the Einstein frame [12–14]. In this case, Eq. (16) simplifies to

$$\frac{dy}{dN}=\frac{3}{2}\frac{[w_k(y)-1]}{r'(y)}\left[r(y)-\sqrt{\frac{\varepsilon_k}{\varepsilon_{tot}}}\right], \quad (20)$$

where we restrict ourselves to the most interesting case of positive σ on the branch of positive φ. To close the system of equations for the two unknown variables y and $\varepsilon_k/\varepsilon_{tot}$, we use the equation

$$\frac{d(\varepsilon_k/\varepsilon_{tot})}{dN}=3\frac{\varepsilon_k}{\varepsilon_{tot}}\left(1-\frac{\varepsilon_k}{\varepsilon_{tot}}\right)[w_m-w_k(y)], \quad (21)$$

which immediately follows from Eq. (9). If y_m is a solution of Eq. (18), then $y(N)=y_m=$const, satisfies Eqs. (20) and (21), provided

$$r^2(y_m)=\left(\frac{\varepsilon_k}{\varepsilon_{tot}}\right)_m<1, \quad (22)$$

where the inequality is simply the physical constraint that $\varepsilon_k<\varepsilon_{tot}$ (assuming positive energy densities ε_k and ε_m). If $r(y_m)>1$, a tracker solution $y(N)=y_m$ is physically forbidden.

A. When are trackers attractors?

To find out when trackers are stable solutions with a nontrivial basin of attraction, we study the behavior of small deviations from the tracker solution. Substituting $y(N)=y_m+\delta y$ and $\varepsilon_k/\varepsilon_{tot}(N)=(\varepsilon_k/\varepsilon_{tot})_m+\delta(\varepsilon_k/\varepsilon_{tot})$ into Eqs. (20) and (21) and linearizing, we obtain

$$\frac{d\,\delta y}{dN}=\frac{3}{2}\frac{[w_k(y_m)-1]}{r'_m}\left[r'_m\delta y-\frac{\delta(\varepsilon_k/\varepsilon_{tot})}{2r_m}\right], \quad (23)$$

$$\frac{d\,\delta(\varepsilon_k/\varepsilon_{tot})}{dN}=-3r_m^2(1-r_m^2)w'_k(y_m)\,\delta y, \quad (24)$$

where the index m denotes evaluation of the appropriate quantities at the tracker point y_m and $(\varepsilon_k/\varepsilon_{tot})_m$ has been replaced by $r^2(y_m)$ according to Eq. (22). Differentiating Eq. (23) with respect to N and using Eq. (24), one obtains the following closed equation for δy:

$$\frac{d^2\delta y}{dN^2}+\frac{3}{2}(1-w_m)\frac{d\,\delta y}{dN}+\frac{9}{2}(1-r_m^2)(1+w_m)(c_S^2-w_m)\,\delta y=0. \quad (25)$$

Here c_S^2 is the squared ''speed of sound'' of *k*-essence at the tracker point and we took into account that $w_k(y_m)=w_m$. Equation (25) is a second order differential equation with constant coefficients and has two exponential solutions. It is easy to see that for $|w_m|<1$ both solutions decay if

$$c_S^2>w_m. \quad (26)$$

Therefore, since $c_S^2=(g-g'y)/g''y^2$, any tracker can be easily made an attractor by arranging a small second derivative of g at the tracker point.

As important examples, let us consider the two most interesting cases, namely, trackers in the presence of radiation (labeled r in the equations below) and cold matter (labeled D for ''dust'').

B. Radiation trackers

For radiation trackers, $w_m \equiv w_r = 1/3$ and Eq. (18), which defines the location of the radiation trackers ($y_m \equiv y_r$), reduces to

$$y_r g'(y_r) = -3g(y_r). \tag{27}$$

The ratio of the energy densities is given by

$$\left(\frac{\varepsilon_k}{\varepsilon_{tot}}\right)_r = r^2(y_r) \equiv -2g'(y_r)y_r^2 \tag{28}$$

and radiation trackers exist only if, at the points y_r satisfying Eq. (27), $r^2(y_r) < 1$. These trackers are stable attractors only if $g''(y_r) < -4g'(y_r)/y_r$. Radiation trackers are always located in the region where $g > 0$ (positive pressure), corresponding to $y < y_D$ in Fig. 1. For a given $g(y)$, there can be more than one radiation tracker. For each of them, the geometrical way of finding the value of y corresponding to the tracker is given in Fig. 1. These trackers can have different values of $r^2(y_r) = (\varepsilon_k/\varepsilon_{tot})_r$. Numerically, a likely range for $r^2(y)$ is 10^{-1}–10^{-2}. This is also the range we wish to have in order that cosmic acceleration begin at roughly the present epoch. We label the radiation tracker with the desired value of $r^2(y_R)$ as **R**, and a second possible radiation tracker with a different value of $r^2(y_r)$ (the one closest to y_D) as **r**(?) in Fig. 1. If $r^2(y_r)$ is much smaller than 10^{-2}, the energy density falls so much at the onset of matter domination (before it freezes at a constant value) that it would not yet have overtaken the matter density today. If $r^2(y_r)$ is much greater than 10^{-1}, then the contribution of *k*-essence to the total energy density would change the expansion rate in the early universe and adversely affect the predictions of primordial nucleosynthesis. The current constraints on $r^2(y_r)$ from nucleosynthesis vary from 4% [16] to 20% [17], depending on how the observations are weighted.

C. Dust trackers

The *k*-essence field can also track the dust ($w_D = 0$) in the (cold) matter-dominated universe. Since the pressure is proportional to $g(y)$ and is zero for dust, it must be that

$$g(y_D) = 0, \tag{29}$$

at the dust attractor point, $y = y_D$. An additional condition for the existence of the dust tracker is that $r(y_D) < 1$ [see discussion following Eq. (22)]. In this case the ratio of energy densities at the dust tracker is given by

$$\left(\frac{\varepsilon_k}{\varepsilon_{tot}}\right)_D = r^2(y_D) = -\frac{9}{8}g'(y_D)y_D^2. \tag{30}$$

If a dust tracker exists, then it is always an attractor, since the stability condition, Eq. (26), just means here that the ''speed of sound'' of *k*-essence should be positive. Note that for the monotonically decreasing convex functions g under consideration only a maximum of one dust attractor can exist (see Fig. 1) since g has only one zero. It is very important to point out that one can easily avoid a dust tracker by considering functions g such that $r^2(y_D) = -\frac{9}{8}g'(y_D)y_D^2 > 1$ at y_D.

D. de Sitter attractors

We have noted that *k*-essence can have attractor solutions which are not trackers in that they do not mimic matter or radiation. These attractor solutions play an important role in two extreme cases, namely, when the energy density of matter or radiation is either much bigger or much smaller than the energy density of *k*-essence. In this subsection, we study the case when the background is dominated by matter radiation and *k*-essence is an insignificant component, $\varepsilon_k \ll \varepsilon_m$. In this case, if $g(y)$ satisfies some simple properties, *k*-essence has an attractor solution in which it behaves like a cosmological constant ($w_k \to -1$). We refer to this solution as the de Sitter attractor (labeled **S**).

Our purpose is to construct models in which *k*-essence has a positive pressure, radiation tracker solution (**R**) during the radiation-dominated phase and approaches a state with negative pressure shortly after the onset of the matter-dominated phase. At the very least, it is necessary that $g(y)$ be positive for some range of y and negative for another range since the pressure is proportional to $g(y)$. This simple condition is generically sufficient to produce a de Sitter attractor solution: Since g' must be negative [the positive energy condition, Eq. (15)], it follows that g must have a unique zero, y_D, the only dust attractor possible. Furthermore, $g(y)$ is positive for $y < y_D$, a range which must include the radiation tracker, $y = y_R$. For $y > y_D$, the pressure ($\propto g$) and, correspondingly, $w_k = -g/yg'$ are negative. From this observation, combined with the stability condition [$g'' > 0$; see Eq. (15)], it follows that the derivative of $r(y)$ [see definition (17)],

$$r' = \frac{3}{4\sqrt{2}}\frac{g''y}{\sqrt{-g'}}(w_k - 1), \tag{31}$$

must be negative for $y > y_D$. Since $r(y)$ is positive at $y = y_D$ and has a negative derivative for $y > y_D$, generically (provided r' does not approach zero too rapidly) $r(y)$ should vanish at some point $y = y_S > y_D$ and then become negative. As immediately follows from the definition of r [see Eq. (17)], the equation of state of *k*-essence at $y = y_S$ (point S in Fig. 1) corresponds to a cosmological term: $w_k(y_S) = -1$. Hence, we see that de Sitter attractors exist for a very wide class of $g(y)$ and are a generic feature of *k*-essence models.

In the absence of matter, $y(N) = y_S = \text{const}$ is not a solution of the equations of motion. However, when matter strongly dominates over *k*-essence ($\varepsilon_k/\varepsilon_{tot} \ll 1$), there exists a solution in the vicinity of this point. [Formally, in the limit $\varepsilon_k/\varepsilon_{tot} \to 0$, $y(N) \to y_S$ is an exact solution of Eqs. (20) and (21).] Setting $w_m = w_k = -1$ in Eq. (23) it can be also verified that this is a stable attractor. For a finite, but very small ratio $\varepsilon_k/\varepsilon_{tot} \ll 1$, the approximate solution, corresponding to $w \approx -1$, is located in the vicinity of y_S and has the form

$$\frac{\varepsilon_k}{\varepsilon_{tot}}(N) \propto \exp[3(1 + w_m)N] \tag{32}$$

and

$$y(N) \approx y_S + \frac{2}{(3+w_m)r'(y_S)} \left(\frac{\varepsilon_k}{\varepsilon_{tot}}(N) \right)^{1/2}. \tag{33}$$

As shown below, if at any moment of time $\varepsilon_k/\varepsilon_{tot}$ lies below the basin of attraction of the tracker solutions, k-essence will be driven first to the de Sitter attractor and stay in its vicinity as long as $\varepsilon_k/\varepsilon_{tot}$ is sufficiently small. We will utilize this property at the transition from the radiation-to the matter-dominated phase.

E. *k*-attractors

Whereas the de Sitter attractors are important when k-essence is an insignificant contribution to the total energy density, the k-attractors arise when k-essence is the dominant energy component. In the absence of matter ($\varepsilon_k/\varepsilon_{tot}=1$), the function $y(N)=y_k=$const, where y_k satisfies the equation

$$r(y_k)=1, \tag{34}$$

is a solution of Eq. (20), while Eq. (21) is satisfied identically. This solution describes a power-law expanding universe [8,10]. The equation of state can be easily obtained from Eqs. (17) and (34),

$$1+w_k(y_k) = \frac{2\sqrt{2}}{3} \frac{1}{\sqrt{-g_k' y_k^2}} = \text{const}, \tag{35}$$

and the scale factor is

$$a \propto t^{2/3(1+w_k)} = t^{\sqrt{-g_k' y_k^2/2}}. \tag{36}$$

If $-g_k' y_k^2/2>1$, the solution describes power law inflation, which is an attractor of the system provided that $r'(y_k)<0$.

The existence of a k-attractor depends mainly on the form of the function $r(y)$. A k-attractor corresponds to $r(y_k)\to 1$ (i.e., the limit where the energy density is totally dominated by k-essence). In general, if $r(y_0)>1$ for some y_0 and there exists an **S**-attractor $[r(y_S)=0]$, then there must exist a k-attractor somewhere between them, $y_0<y_k<y_S$, simply because $r(y)$ is a continuous function.

In particular, we are interested in the case where there is no dust attractor because $r(y_D)>1$, and yet there is a de Sitter attractor with $r(y_S)=0$. In this case, not only must there exist a k-attractor at some $y_D<y_K<y_S$, but we know that it has *negative pressure* [since $g(y_K)<0$], is *stable* [since $w_k<1$; see Eq. (31)] and is the *unique* k-attractor with negative pressure (since r' is monotonically decreasing in this y interval).

Note also that this negative-pressure k-attractor only exists if there is no dust tracker solution, that is, $r(y_D)>1$. If there is a dust tracker $[r(y_D)<1]$, then, since $r'(y)<0$ for $y>y_D$, there is no point $y=y_K>y_D$ where $r(y)=1$ and, hence, there is no k-attractor at $y_D<y<y_S$.

It is possible to have other k-attractors with positive pressure at $y<y_D$ [the closest one to y_D is denoted by **k**(?) in Fig. 1], but they will prove to be irrelevant in our scenario.

IV. COSMIC EVOLUTION AND ATTRACTOR SOLUTIONS

Once all possible attractors for k-essence have been identified, it is easy to understand the evolution of the k-field as a voyage from one attractor solution to another as different phases of cosmic evolution proceed. For both the radiation- and matter-dominated phases, there are several possible configurations of relevant attractor solutions. In this section, we systematically classify the attractor configurations for each phase and their consequences for cosmic evolution.

A. Radiation domination

We assume that $g(y)$ has been chosen so that there exists an attractor solution (**R**) at $y=y_R$ such that $r^2(y_R) \equiv \varepsilon_k/\varepsilon_{tot}$ is in the range 1–10 %, roughly equipartition conditions. This energy ratio leads most naturally to a matter-dominated epoch that lasts a few 10^9's of years and cosmic acceleration beginning at about the present epoch. Depending on the form of $r^2(y)$, which is determined by $g(y)$ in the Lagrangian, there will be additional attractors during the radiation epoch. Whether y is drawn to the correct attractor y_R depends on initial conditions and the other attractors. Ideally, we want $y=y_R$ to have the largest basin of attraction so that most initial conditions join onto the desired cosmic track. The combination of cosmologically relevant attractors during the radiation-dominated phase can be one of three types:

(A_r) **R**, **S** and *no* other attractors at $y_S>y>y_R$. This occurs only if the function $r(y)$ decreases for $y_R<y<y_S$. Conversely, if $r(y)$ increases somewhere in the range $y>y_R$, then it inevitably leads to the appearance of an extra k- and/or r-attractor at $y>y_R$. Let us prove it.

If the function $r(y)$ increases within some interval, it means that the derivative $r'(y)$ is positive there. On the other hand, as it follows from Eq. (31), $r'(y)$ is positive only if $w_k>1$. Since $w_k(y_R)=1/3$, $w_k(y_S)=-1$ and $w_k(y)>1$ somewhere in the interval $y_R<y<y_D$, there must be another point $\overline{y}$ within this interval, where $w_k(\overline{y})=1/3$. If $r(\overline{y})<1$, this point is a radiation tracker different from **R** with a different value of $r^2(y)$. If $r^2(\overline{y})>1$, then $\overline{y}$ is not a tracker at all, but since $r(y_S)=-1$, there must exist a point in the interval $y_S>y_k>\overline{y}$ where $r(y_k)=1$, which corresponds to a k-attractor. That is, either there is a an extra radiation tracker or there is an extra k-attractor.

For models of type (A_r) where $r^2(y)$ is monotonically decreasing, a dust tracker solution with $r(y_D)<r(y_R)$ is inevitable and k-essence will be attracted immediately to it after matter-radiation equality, a situation we are trying to avoid in order to explain the present-day cosmic acceleration. The model $\tilde{p}(X)=-1+X$ falls in the above category; with a field redefinition, the action can be recast into the model of a field with canonical kinetic energy rolling down an exponential potential [4], an example which is well

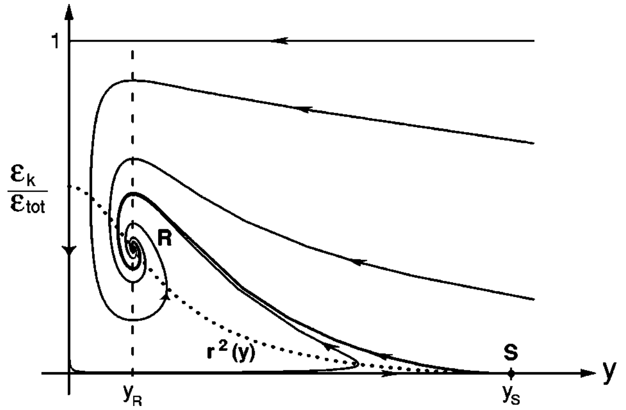

FIG. 2. Phase diagram for case (A_r) during the radiation-dominated epoch. Phase lines flow in the direction shown by the arrows, dashed horizontal lines determine the y coordinate of attractor solutions and boldface labels the corresponding attractor points. The dotted line shows the points where $\varepsilon_k/\varepsilon_{tot}=r^2(y)$.

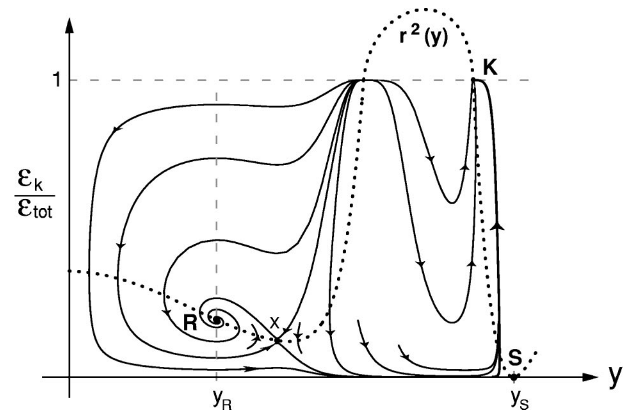

FIG. 3. Phase diagram of a model of the type (B_r) during the radiation-dominated phase. In the relevant region of the diagram all trajectories can be traced back to a common origin. Some of the phase trajectories converge to the radiation tracker **R**, while others, after approaching the de Sitter point **S**, finally reach the **K**-attractor. The saddle point **x** ''separates'' both types of trajectories.

known to track in both the radiation- and matter-dominated epochs.

(B_r) **R**, **S**, **K** plus possibly other attractors at $y<y_D$. This situation takes place when there is no dust tracker solution $[r(y_D)>1]$ the case considered in our first paper [3].

(C_r) **R**, **S** (*no* **K** attractor) and at least one additional attractor **r**(?) or **k**(?). This case occurs whenever there is a dust tracker solution $[r(y_D)<1)]$ with the property that $r(y_D)>r(y_R)$ or, in other words, $(\varepsilon_k/\varepsilon_{tot})_D>(\varepsilon_k/\varepsilon_{tot})_R$. Even though there exists a dust tracker solution, we will show it is nevertheless possible to have a finite period of cosmic acceleration at the present epoch before k-essence reaches the dust tracker solution in the future. For this to occur, the function $r(y)$ must increase somewhere in the interval $y_R<y<y_D$. This is precisely the case considered above [see discussion of case $(A)_r$], where we argued that there must be an extra r- and/or k-attractor in the interval $y_D<y<y_R$. Furthermore, the attractor closest to y_D must have $r(y_{r/k})>r(y_D)>r(y_R)$; otherwise, we could find another attractor in the interval $y_{r/k}<y<y_k$, as can be shown by repeating the argument presented under A_r for this interval. If $r(y_{r/k})>r(y_D)>r(y_R)$, this second tracker has a larger fraction of k-essence.

A phase diagram of the system of Eqs. (20),(21) describing the global evolution of the k-field during radiation domination is shown in Figs. 2, 3 and 4 for each of cases (A_r), (B_r) and (C_r) respectively. Phase trajectories cannot cross the lines where $\varepsilon_k/\varepsilon_{tot}$ is equal to 0 or 1, and, hence, their tangents are horizontal there. The position of the radiation tracker **R** is fixed by the intersection of the $y=y_R$ line (dashed line) and the $r^2(y)$ curve (dotted line). If $r^2(y)$ is bigger than 1 at the intersection point, the tracker does not exist. Notice that the phase trajectories go in the direction of increasing (decreasing) $\varepsilon_k/\varepsilon_{tot}$ for $w_k(y)<1/3$ $[w_k(y)>1/3]$ and, therefore, their tangents are horizontal at the points where $w_k(y)=1/3$. On the other hand, phase trajectories evolve in the direction of increasing [decreasing] y for $\varepsilon_k/\varepsilon_{tot}<r^2(y)[\varepsilon_k/\varepsilon_{tot}>r^2(y)]$ and at the points where these phase lines cross the curve $r^2(y)$ their tangents are horizontal [see Eq. (10)]. The form of $r(y)$ also gives a clue about the equation of state $w_k(y)$: in the region where $r(y)$ is an increasing function of y we have $w_k(y)>1$ and where it decreases $w_k(y)<1$. Hence, as noted previously, $r(y)$ is what mainly determines the structure of the phase diagram.

As clearly seen in the figures in all cases, if the k-field is initially located near the **R**-tracker, it converges to it. Therefore, the basin of attraction is non-zero in all three cases. The attraction region includes equipartition initial conditions, the most natural possibility.

For (A_r), Fig. 2, the **R**-attractor has the largest basin of attraction, the complete phase plane. If one starts, for instance, at $(\varepsilon_k/\varepsilon_{tot})_i=\exp(-30)(\varepsilon_k/\varepsilon_{tot})_R$, then the k-field rapidly reaches the vicinity of the de Sitter point **S** and joins the attractor connecting this point to the **R**-tracker.

Cases (B_r) and (C_r) have limited basins of attraction, and so are not as favorable from the point of view of initial conditions. If the energy density of the k-field is much smaller than the value at the **R**-tracker, the k-field travels first to the vicinity of the **S**-attractor, where it meets the phase trajectory that connects it to the **K**-attractor [case (B_r)] or the **r**-attractor [case (C_r)]. In either situation, the field never

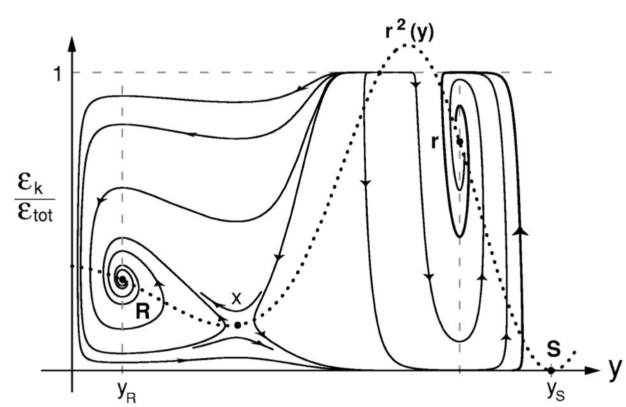

FIG. 4. Phase diagram of a model of the type (C_r) during radiation domination, with same notation as in Fig. 3.

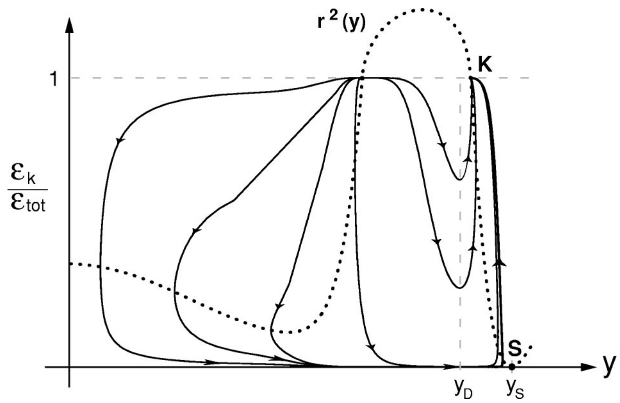

FIG. 5. Phase diagram of a model of type (A_d) during the matter-dominated epoch. All trajectories have a common origin and all of them finally reach the **K**-tracker. Trajectories which ''skim'' the line $\varepsilon_k/\varepsilon_{tot}\approx 0$ reach this attractor after going through a nearly de Sitter stage (the **S**-attractor).

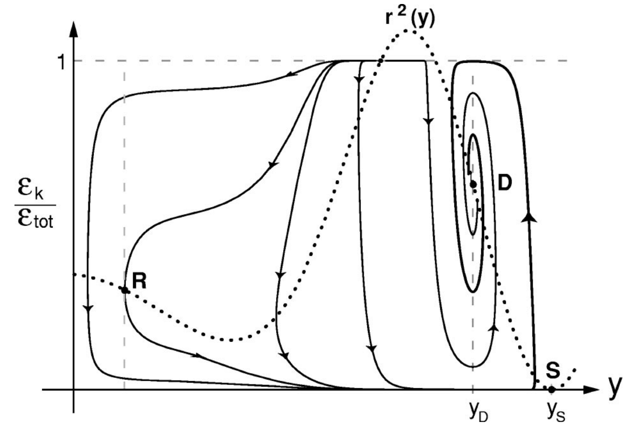

FIG. 6. Phase diagram of a model of type (B_d) during the matter-dominated epoch. All trajectories have a common origin and all of them finally reach the **D**-tracker. Trajectories which ''skim'' the line $\varepsilon_k/\varepsilon_{tot}\approx 0$ reach this attractor after going through a nearly de Sitter stage.

reaches the **R**-tracker. Although the latter two cases have smaller basins of attraction than case (A_r), only cases (B_r) and (C_r) can produce cosmic acceleration today. One can simply assume that the initial value of the *k*-field lies somewhere in the basin of attraction, a reasonable possibility. An alternative is to introduce additional φ dependence in the Lagrangian, as for instance, $L=g(y,\varphi)/y\varphi^2$, where $g(y,\varphi)\to g_1(y)$ at high energies (φ is smaller than some φ_0) and $g(y,\varphi)\to g_2(y)$ at relatively low energies (φ is bigger than φ_0), such that $g_1(y)$ has an (A_r) set of attractors and $g_2(y)$ has a (B_r)/(C_r) set of attractors. Note that the exact value of φ_0 is not important at all; we only have to be sure that the transition from one regime to the other happens before equipartition. Although modifying the Lagrangian may seem more complicated, it has the advantage that it removes nearly altogether dependence on initial conditions.

B. Matter domination

We have shown that it is possible to choose a wide range of models and initial conditions for which the *k*-field converges to the **R**-tracker during the radiation-dominated epoch. The goal is to produce a scenario in which *k*-essence overtakes the matter density and induces cosmic acceleration today. Yet the contribution of *k*-essence to the total energy density must not spoil big bang nucleosynthesis or dominate over the matter density at the end of the radiation-dominated epoch (see Sec. III B). To satisfy these conditions, it typically suffices if the **R**-tracker satisfies

$$(\varepsilon_k/\varepsilon_{tot})_R=r^2(y_R)=\alpha\simeq 10^{-2}\text{–}10^{-1}. \tag{37}$$

In this subsection, we study the evolution as the universe enters the matter-dominated epoch and the *k*-field is forced to leave the radiation tracker. In a dust dominated epoch the relevant attractors can appear in the following two possible sets: (A_d) **S**, **K** and (B_d) **S**, **D**.

In both cases successful *k*-essence models are possible. In case (A_d), which was discussed in our earlier paper [8], there is no dust tracker solution [$r(y_D)>1$]. Therefore, as seen in the phase diagram of Fig. 5, when the radiation-dominated epoch is over, *k*-essence approaches first the **S**-attractor; afterwards, when its energy density has increased significantly, it moves to the **K**-attractor (a state with negative pressure but $w_k>-1$). If $w_k(y_K)<-1/3$, the expansion rate accelerates for the indefinite future; if $-1/3<w_k(y_K)<0$, the expansion rate decelerates. Either way, the matter-radiation density is increasingly negligible compared to *k*-essence in the far future.

In the second case (B_d), there is a dust tracker solution. If $(\varepsilon_k/\varepsilon_{tot})_D\ll 1$, *k*-essence contributes only a small fraction of the total energy density at this attractor, and it approaches this attractor almost immediately after matter-radiation equality. This is not desirable since then *k*-essence cannot dominate today or cause cosmic acceleration. However, if $(\varepsilon_k/\varepsilon_{tot})_D=r^2(y_D)\to 1$ or $(\varepsilon_k/\varepsilon_d)_D\gg 1$, there can be a period of cosmic acceleration before the *k*-field reaches the dust attractor since it can first approach the **S**-attractor and remain there for a finite time; see Fig. 6. Ultimately, though, the acceleration is temporary; the *k*-field proceeds to the dust tracker, the expansion of the universe begins to decelerate, and the ordinary and (cold) dark matter density approaches a fixed, finite fraction of the total energy. We refer to the scenario as a ''late dust tracker'' because the dust attractor is reached long after matter-domination has begun.

Taking into account that $r(y_D)$ is near unity or greater for both cases (A_d) and (B_d), we obtain, from Eqs. (17) and (37),

$$\frac{g_R' y_R^2}{g_D' y_D^2}\leqslant\frac{9}{16}\alpha\simeq 5\times(10^{-3}\text{–}10^{-2}). \tag{38}$$

We can also infer from Fig. 1 that $g_D'(y_R-y_D)\leqslant g(y_R)=-y_R g_R'/3$ and, therefore, for $\alpha\ll 1$,

$$\frac{y_R}{y_D}\leqslant\frac{3}{16}\alpha\simeq 2\times(10^{-3}\text{–}10^{-2}) \tag{39}$$

and

$$\frac{g'_D}{g'_R} \leqslant \frac{\alpha}{16} \simeq 6\times(10^{-4}\text{–}10^{-3}). \tag{40}$$

Since $\varepsilon_k = -g'/\varphi^2$ and $|g'(y_S)| \leqslant |g'(y_D)|$, we conclude that after radiation domination, when the *k*-field reaches the vicinity of the **S**-attractor, the ratio of energy densities in *k*-essence and dust cannot exceed $\varepsilon_k/\varepsilon_d < \alpha^2/16 \simeq 6 \times(10^{-6}\text{–}10^{-4})$. This is the nadir of *k*-essence; once *k*-essence approaches the **S**-attractor, its contribution to the cosmic density increases again until it becomes comparable to the matter density. In case (A_d), the *k*-field will evolve further to the **K**-attractor and the *k*-essence energy will increasingly dominate over the matter density. In case (B_d), the *k*-field approaches the **D**-tracker where the ratio of *k*-essence to the matter density approaches some fixed positive value.

The statements above are generic and do not depend significantly on the concrete model as long as it satisfies the simple criteria formulated above. Let us stress that the only ''small'' parameter used is the ratio $(\varepsilon_k/\varepsilon_{tot})_R$, which has to be of the order of $10^{-2}\text{–}10^{-1}$, a very natural range for these models and one that satisfies constraints of big bang nucleosynthesis (see Sec. III B). For this range, the present moment is approximately the earliest possible time when cosmic acceleration could occur.

Finally note that, during the transition from the radiation tracker **R** to the de Sitter attractor **S**, the equation of state of *k*-essence has to take values bigger than 1, and hence the dominant energy condition $\varepsilon_k > |p_k|$ is violated during a certain finite time interval. This violation implies that *k*-essence energy can travel with superluminal speeds [15]. Thus, perfectly Lorentz-invariant theories containing non-standard kinetic terms seem to allow the presence of superluminal speeds, as already pointed out in [10,18].

V. CONSTRUCTING MODELS

In previous sections, we have presented a general theoretical treatment of the attractor behavior of *k*-essence fields in a cosmological background. We have emphasized the properties needed to formulate models which will lead naturally to cosmic acceleration at the present epoch. In this section, we discuss how to apply the general principles to construct illustrative toy models.

Let us summarize the conditions we have derived for building viable Lagrangians. First, we must satisfy the general positive energy and stability conditions in Eq. (15). If g takes positive and negative values, they already suffice to guarantee generically the existence of a radiation point y_R where $w(y_R)=1/3$, a unique dust point y_D where $w(y_D)=0$, and a unique de Sitter point y_S where $w(y_S)=-1$. The radiation point is an attractor if $g''(y_R)$ is sufficiently small,

$$g''(y_R) < -4\frac{g'(y_R)}{y_R}, \tag{41}$$

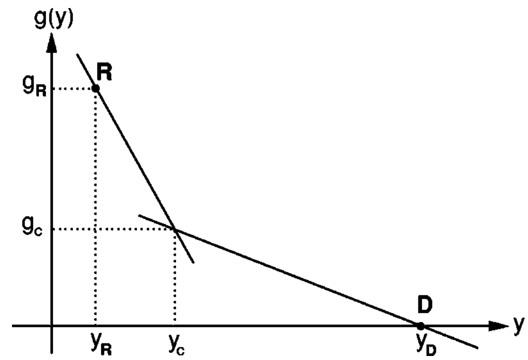

FIG. 7. A simple toy model for $g(y)$ consisting of two linear pieces meeting at the ''crossing point'' y_c. Here y_R and y_D are the radiation and the dust attractor values, and the derivatives of g at these points are g'_R and g'_D, respectively.

and the remaining prerequisites needed to ensure a successful scenario are then reduced to simple restrictions on the derivative of g at two separate values of y:

(i) At y_R, $r_R^2 = -2g'(y_R)y_R^2 \simeq 10^{-2}\text{–}10^{-1}$.

(ii) At y_D either $r_D^2 = -9y_D^2 g'(y_D)/8 > 1$ or $1-r_D^2 = 1+9y_D^2 g'(y_D)/8 \ll 1$.

The first condition in (ii) corresponds to cases where there is no dust attractor, and the second condition to cases where there is a dust attractor with a small matter to *k*-essence energy density ratio.

A straightforward way of constructing a function with given derivatives at two points is to glue two linear functions with the required slopes, as shown in Fig. 7. Observe that if $g(y)$ is linear around the radiation point, the attractor requirement (41) is automatically satisfied. In order to have a finite c_S^2, it suffices to introduce small quadratic corrections to the glued linear functions. We implement this procedure to build a toy model expressed in terms of artificial parameters (from the point of view of fundamental physics) that can be simply related to Fig. 7 and our earlier discussion of attractor solutions. One should appreciate that, for this pedagogical purpose, we have ''overparametrized'' the problem—the outcome is rather insensitive to most parameters as long as they obey certain simple general conditions. Simpler forms with fewer parameters are certainly possible.

Let $g_{glue}(y)$ be any smooth function constructed by gluing the two linear pieces of Fig. 7. The function g_{glue} depends on y and has y_R, g'_R, y_D and g'_D as parameters where y_R and y_D are the radiation and the dust attractor values and the derivatives of g at these points are g'_R and g'_D respectively. Our toy model corresponds to

$$g(y) \equiv g_{glue}(y)\left(1-\frac{y}{s^2 y_D}\right). \tag{42}$$

The factor g_{glue} describes the function in Fig. 7 and the factor in parentheses provides the quadratic corrections needed to have a positive speed of sound. It so happens that the latter factor also shifts the de Sitter point from $y=\infty$, as it would be for purely linear functions, to finite y, although this is not crucial for our purpose. For $s \gg 1$ the de Sitter point is located at $y_S \approx s y_D$ and $g \approx g_{glue}$.

Once a general form for g is known, such as the example above, one can study how the model parameters affect the resulting cosmology. Our conclusion is that the predictions of the toy model are relatively insensitive to the gluing function or to the particular values of y_R, y_D, g_R', g_D' and y_S as long as they satisfy certain simple relations. For instance, what sets the values of Ω_k and w_k today? Do these depend on the precise form of the interpolating function? We have solved numerically the equations of motion for a wide range of gluing functions g_{glue} in Eq. (42). For a typical parameter choice, the final value of Ω_k does not depend on the particular gluing function as long as g_{glue} conforms closely enough to Fig. 7.

The value of Ω_k today does depend on the evolution of $\varepsilon_k/\varepsilon_m$. At early times the field is locked at the radiation tracker, and its fractional energy density ratio is given by $-2g_R' y_R^2$. After radiation-matter equality the field cannot follow the radiation tracker anymore and its energy density drops by several orders of magnitude until $\varepsilon_k/\varepsilon_m$ reaches a minimum value at the time w_k falls below zero. We shall label this minimum value with the subscript ''min.'' The energy density at this minimum is roughly given by

$$\left(\frac{\varepsilon_k}{\varepsilon_m}\right)_{min} \approx r_R^2 \frac{g_D'}{g_R'}. \tag{43}$$

The position of the minimum in time only depends on the distance between the radiation and crossing point $y_c - y_R$. As $y_c - y_R$ increases from zero, the minimum is shifted from matter-radiation equality to later times. After reaching the minimum, the field moves onto the de Sitter attractor and $\varepsilon_k/\varepsilon_m$ grows as $(z+1)^{-3}$, where z is the redshift. In order to have k-essence dominate today, it must be that $\varepsilon_k/\varepsilon_m$ during the radiation epoch lies roughly between 10^{-1} and 10^{-2}. Then, $(\varepsilon_k/\varepsilon_m)_{min}$ lies in the range 10^{-4}–10^{-6} and, provided y_c is chosen appropriately, this has k-essence dominating at about the present epoch. One can see these conditions impose constraints on certain combinations of our parameters, although in a fairly natural range not very far from unity.

As discussed in Sec. IV B, there are two possible future fates for the universe depending upon whether there is a ''late dust tracker'' solution or not. By requiring $r_D^2 > 1$ we avoid a dust tracker and, therefore, ensure that the k-field approaches the k-attractor when k-essence starts to dominate. The equation of state of k-essence at the k-attractor depends on the parameter s. By increasing s the equation of state w_k at the k-attractor approaches -1, and in the limit $s \to \infty$, $w_k(y_K) \to -1$. If $w_k < -1/3$, the expansion rate of the universe accelerates forever. Using the maximal value of the w at the present epoch as allowed by supernova observations, say, s can be simply adjusted to ensure that w at the k-attractor is less than or comparable to this value. In this case, the equation of state of k-essence today will be less than or equal to $w_k(y_K)$, which is set by s, as described above.

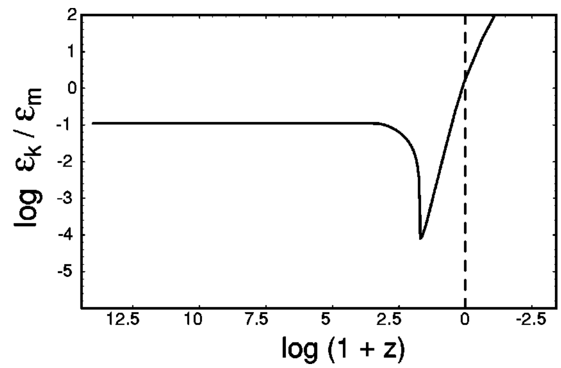

FIG. 8. The ratio of k-essence to matter energy density, $\varepsilon_k/\varepsilon_m$, vs $1+z$ for a model with a k-attractor.

If $r_D^2 < 1$, it is possible to have successful models if r_D^2 is sufficiently close to 1. In such a model the equation of state of k-essence will finally reach $w_k = 0$ in the far future; so, ultimately, cosmic acceleration ceases and the expansion begins to decelerate again. Nevertheless, it is still possible to have a finite period in which the equation of state is negative and which includes the present epoch. It is worth noting that models without a dust attractor are more generic and natural, since they do not require a special tuning of $r(y_D)$ to a value close but smaller than unity at the dust point. Below we illustrate examples of both types.

A. Model without a dust attractor

Models that belong to the general class (A_d) illustrated in Fig. 5 do not have dust attractor solutions because $r(y_D) > 1$. Choosing the following values of the parameters, $y_R = 0.1$, $g_R' = -5$, $y_D = 17$, $g_D' = -5 \times 10^{-3}$ and $s^2 y_D = 135$, we have $r(y_D) \approx 1.2$. Therefore, there has to be a **K**-inflationary attractor, which is located for our parameter choice at $y_K \approx 28$. At the **K**-attractor, k-essence has the equation of state $w_k(y_K) \simeq -0.43$. The ratio of the energy densities at the **R**-tracker in this model is $(\varepsilon_k/\varepsilon_{tot})_R = 0.1$. The results of the numerical calculations are presented in Figs. 8 and 9. We see that during the radiation stage k-essence quickly reaches the radiation tracker, in particular, the oscillations of the equation of state w_k in Fig. 9 around $w_k = 1/3$ decay exponentially rapidly. The k-field has the same equation of state as radiation until the moment when dust starts to

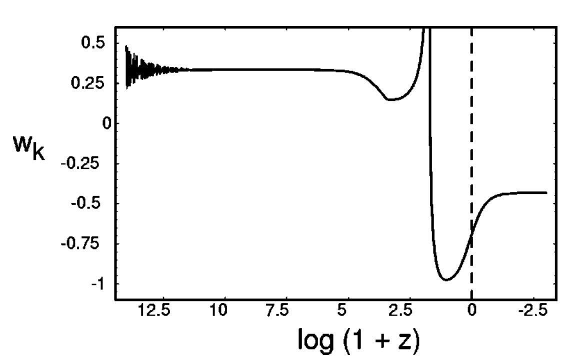

FIG. 9. The equation of state w_k vs $1+z$ for a model with a k-attractor.

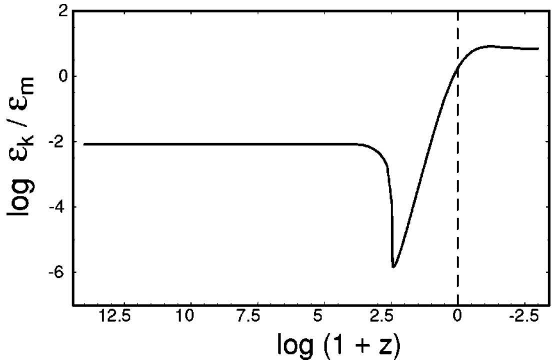

FIG. 10. The ratio of k-essence to matter energy density, $\varepsilon_k/\varepsilon_m$, vs $1+z$ for a model with a late dust tracker solution. In this type of model, $w_k \rightarrow 0$ in the far future and the ratio of k-essence to matter energy density approaches a constant.

dominate. Around this time the energy density of k-essence suddenly drops by three orders of magnitude and the equation of state, after a very short period of increase, drops down to $w_k \simeq -1$, the value of the equation of state along the **S**-attractor. After that, when the energy density of k-essence becomes significant, w_k starts to increase towards the **K**-attractor value, -0.43. Since Ω_k is not yet unity, the current value is somewhere between the **K**-attractor value and -1; in this example, the value today $(z=0)$ is $w_k \simeq -0.69$. The energy density of k-essence today is $\Omega_k \approx 0.65$, and because we assumed a flat universe, $\Omega_m = 0.35$. For completeness let us mention that we have defined ''today'' $(z=0)$ to be the moment when the matter-radiation energy density ratio is given by $(\varepsilon_r/\varepsilon_m)_{today} \equiv 4.307 \times 10^{-5}/(\Omega_m h^2)$.

B. Model with a late dust attractor

Taking $y_R = 11 \times 10^{-3}$, $g_R' = -34$, $y_D = 11$, $g_D' = -8 \times 10^{-3}$ and $s^2 y_D = 56$, we can construct a model with a ''late dust tracker,'' corresponding to the phase diagram in Fig. 6. The parameters have been deliberately chosen to differ significantly from the ones in the model without dust attractor in order to illustrate that fine-tuning is not necessary.

The late dust attractor is reached after k-essence passes near the de Sitter attractor following matter-radiation equality. At the late dust tracker $(\varepsilon_k/\varepsilon_{tot})_D = r^2(y_D) \simeq 0.88$ and, correspondingly, $(\varepsilon_k/\varepsilon_d)_D \simeq 7$. Hence, the fractional contribution of the matter density is small but remains finite in the indefinite future. The ratio of energies at the **R**-tracker is $(\varepsilon_k/\varepsilon_{tot})_R \simeq 8.3 \times 10^{-3}$. The results of the numerical calculations are presented in Figs. 10 and 11. The evolution of the k-field here is very similar to the one we described in the previous case; the differences between both models occur at small redshifts. The fraction of the critical energy density of k-essence today is in this model also $\Omega_k = 0.65$ and the equation of state w_k takes the value -0.4. The future evolution of the model with a late dust attractor is completely different from what we found in the previous one. Here the ratio of the energy densities of k-essence and dust will continue growing in the future only until it becomes approximately 7. After that it will start to oscillate around this value with exponentially decaying amplitude while the pressure approaches the dust point, where $w_k = 0$.

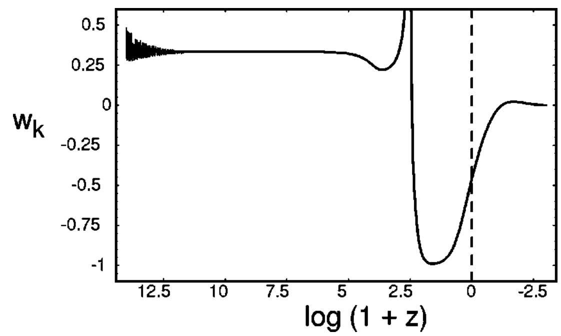

FIG. 11. The equation of state w_k vs $1+z$ for a model with a late dust tracker solution.

C. Simpler and more practical examples

The toy models presented thus far are all built on the ansatz shown in Fig. 7, which entails numerous parameters. We have pointed out that the large number of parameters is not a necessary feature. We have introduced this form for pedagogical purposes, since it enables one to study directly the relation between the attractor solutions and cosmic evolution. Indeed, our analysis showed that the cosmological solution is relatively insensitive to most of the parameters provided they obey a few broad conditions.

To emphasize the point, consider a model of the form

$$\tilde{p}(X) = -b + 2\sqrt{1 + X h(aX)}, \tag{44}$$

where $h(aX)$ is some smooth function that can be expanded in a power series in X. This particular form is reminiscent of a Born-Infeld action in which $h(aX)$ could represent higher order corrections in X. (This choice of a square-root form is not essential—simply an example.) As a specific case, for $b = -2.05$ and $X\, h(aX) = X - (aX)^2 + (aX)^3 - (aX)^4 + (aX)^5 - (aX/2)^6$ the Lagrangian defined by Eq. (44) satisfies all constraints and produces $\Omega_m = 0.3$ and $w_k = -0.8$ today if one chooses $a = 10^{-4}$. This particular example has a cosmic evolution similar to the one described in Sec. V A (no dust attractor). We see that in this case, as with a wide range of other functional forms, the condition $b > 2$ and the choice of the single parameter a suffices to satisfy all of the conditions of the multi-parameter toy models.

VI. DISCUSSION

Introducing a dark energy component with negative pressure has resolved many observational problems with the standard cold dark matter model including the recent evidence from supernova searches that the universe is undergoing cosmic acceleration. At the same time, the dark energy component presents a profound challenge to cosmology and fundamental physics. What is its composition and why has it become an important contribution to the energy density of the universe only recently?

The example of k-essence shows that it is possible to find a predictive, dynamical explanation that does not rely on coincidence or the anthropic principle. Unlike a cosmological constant or quintessence models of the past, the energy density today is not fixed by finely tuning the vacuum density or other model parameters. Rather, the energy density today is forced to be comparable to the matter density today because of the dynamical interaction between the *k*-essence field and the cosmological background.

Technically, the *k*-essence approach, at least in the examples we have constructed, relies on attractor properties that naturally arise if the action contains terms that depend non-linearly on the gradients of the *k*-essence field. Non-linear terms of this type appear in most models unifying gravity with other particle forces, including supergravity and superstring models. In the past, these contributions have been ignored for reasons of ''simplicity.'' The example of *k*-essence demonstrates that the effects of non-linear dynamics can be dramatic. In a cosmological setting, we have shown how they can cause the *k*-essence field to transform from a tracking background field during a radiation-dominated epoch into a an effective cosmological constant at the onset of matter-domination. This effect explains naturally why cosmic acceleration could begin only at low temperatures, at roughly the present epoch.

The non-linear dynamics is totally missed if the kinetic energy terms are truncated at the lowest order contributions. Hence, the kinds of attractor effects discussed in this paper have gone unnoticed in most treatments of quantum field theory. This was one of the reasons for providing a detailed, pedagogical treatment for at least one class of models. Clearly, this is the tip of a broad arena of study. As another possible application, it is interesting to note that a fundamental problem of superstring models is to control the behavior of the many moduli fields in the theory, which are coupled to one another through non-linear kinetic energy terms. At the linear level, the moduli appear to be free fields with a flat potential, and so there is no guidance as to why, among all the possible limits of M theory, the low energy limit looks like the standard model. Perhaps non-linear attractor behavior constrains the evolution of moduli fields.

In this paper, we have focused on how non-linear dynamics addresses a fundamental theoretical issue, the cosmic coincidence problem. An important question to consider is whether there are observational tests to distinguish *k*-essence from alternative explanations. One notable feature of *k*-essence models compared to the more general tracker quintessence models [5,6] is that the equation of state w_k is increasing at the present epoch. For quintessence scalar fields rolling down tracker potentials, the quintessence tracks the matter density ($w=0$) during most of the matter-dominated epoch, and only recently has begun to decrease towards $w=-1$. Hence, measurements of dw/dz for the dark energy would distinguish these two possibilities from one another and from a cosmological constant. However, this test would not distinguish *k*-essence from more general quintessence models that can also be tuned so that w_k is increasing today as well. A second feature of *k*-essence is the non-linear kinetic energy contribution. A consequence is that the effective sound speed c_S^2 is generically different from unity, whereas $c_s=1$ for a scalar field rolling down a potential. Depending on the model, the distinctive sound speed can have subtle or significant effects on the cosmic microwave background anisotropy. We will address these observational considerations in a forthcoming paper [19].

As regards the future of the universe, our work here offers a new, perhaps pleasing possibility. In previous models with a cosmological constant or quintessence, the acceleration of the universe continues forever and ordinary matter that composes stars, planets and life as we know it becomes a rapidly shrinking fraction of the energy density of the universe. In the ''late dust tracker'' scenario which we have introduced here, the acceleration is temporary and the matter density approaches a fixed, finite fraction of the total.

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